Shadow of Bumblebee Black Hole and Newmann Janis

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1 Introduction

Black holes occupy a central place in contemporary astrophysics and gravitational theory, not only for their extreme physical properties but also for the profound insights they offer into the nature of spacetime. For much of the past century, however, the inability to directly observe these compact objects has limited our understanding to indirect evidence and theoretical predictions. This landscape changed dramatically when the Event Horizon Telescope (EHT) Collaboration released the first resolved image of a supermassive black hole at the core of the giant elliptical galaxy M87 [1, 2, 3, 4, 5, 6]. The observation revealed the characteristic shadow of M87*, providing a direct visual signature of strong-gravity effects and marking a transformative milestone in black hole research.

The prospect of Lorentz symmetry breaking in nature has gained considerable attention, largely because several candidate theories of quantum gravity allow for such deviations. In particular, string theory [13, 9], noncommutative field theories [7], and loop quantum gravity [11] all provide mechanisms through which Lorentz invariance may be modified or violated at fundamental scales. This makes the search for observational or experimental traces of Lorentz violation a promising pathway toward uncovering hints of a deeper quantum-gravity framework operating near the Planck scale.

In this paper, we employ the set of observables proposed in Ref. [12] to examine the apparent shadow produced by a rotating black hole in Bumblebee gravity, and to compare it with the shadow associated with the corresponding naked singularity configuration. As a representative example, we assume that the spacetime in the vicinity of the supermassive object Sgr A* is described by the rotating Bumblebee black hole metric, while at sufficiently large distances it smoothly approaches the expected asymptotically flat background.

2 The Schwarzschild-like solution

Bumblebee gravity is a class of vector-tensor theories in which a vector field B_μ acquires a nonzero vacuum expectation value, spontaneously breaking Lorentz symmetry. Casana et al.[8] derived a static, spherically symmetric black hole solution in this framework, modifying the Schwarzschild metric through a Lorentz-violating parameter l that alters both g_{tt} and g_{rr} components:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + (1+l)\frac{1}{1-\frac{2M}{r}} dr^2 + r^2 d\Omega^2 \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

This solution has been widely adopted as a starting point to explore deviations from general relativity and test Lorentz symmetry breaking in strong-field regimes. It has been extended to wormhole geometries, non-linear electrodynamics, and Gauss–Bonnet-type gravity. In order to perform

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Newmann Janis algorithm on above mentioned metric. We first need to goto Eddington-Finkelstein like coordinate. The null geodesic for the transformation to Eddington-Finkelstein coordinate can be found as follows:

$$ds^2 = 0 \implies \frac{dr}{dt} = \frac{1}{1+l}\left(1 - \frac{2M}{r}\right)$$

The advanced Eddington-Finkelstein is given by the transformation $dv = dt - dr^*$.

$$\begin{aligned} dr^* &= \sqrt{(1+l)}\left(1-\frac{2M}{r}\right)^{-1} dr \\ ds^2 &= -\left(1-\frac{2M}{r}\right) dv - 2\sqrt{1+l} dv dr + r^2 d\Omega^2 \end{aligned}$$

We can check that v is indeed null coordinate i.e. $dv^2 = 0$ in following manner:

$$dv_i = \left(1, -\sqrt{(1+l)}\left(1-\frac{2M}{r}\right)^{-1}, 0, 0\right)$$

$$dv^i dv_i = \frac{1}{g_{tt}} \times 1 + \frac{1}{g_{rr}}\left[-\sqrt{1+l}\left(1-\frac{2M}{r}\right)^{-1} \times -\sqrt{1+l}\left(1-\frac{2M}{r}\right)^{-1}\right] = 0$$

The metric tensor in the Eddingten Finkelstein coordinate could be expressed in matrix form to illuminate the difference with Schwarzschild case:

$$g_{\mu\nu} = \begin{vmatrix} -\left(1-\frac{2M}{r}\right) & -\sqrt{(1+l)} & 0 & 0 \\ -\sqrt{(1+l)} & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix}$$

and the corresponding inverse metric looks like:

$$g^{\mu\nu} = \begin{vmatrix} 0 & -\frac{1}{\sqrt{(1+l)}} & 0 & 0 \\ -\frac{1}{\sqrt{(1+l)}} & \frac{1}{1+l}\left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{vmatrix}$$

Next step in this process is to find the null tetrads which could be used to decompose this metric. Based on the choice of normalization $l \cdot n = 1$ and $m \cdot \bar{m} = -1$,we have:

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu$$

In advanced Eddington-Finkelstein coordinate, the null geodesic is directed towards ∂_v, thus let us take that as l^μ and assuming $m_v = m_r = 0$. Thus, the null tetrad can be calculated from the metric tensor in this coordinate and is given by

$$\begin{aligned} l^\mu &= \left(0,\ \frac{1}{\sqrt{1+l}},\ 0,\ 0\right), \qquad & n^\mu &= \left(1,\ -\frac{1}{2\sqrt{1+l}}\left(1-\frac{2M}{r}\right),\ 0,\ 0\right), \\ m^\mu &= \frac{1}{\sqrt{2}\,r}\left(0,\ 0,\ 1,\ \frac{i}{\sin\theta}\right), \qquad & \bar{m}^\mu &= \frac{1}{\sqrt{2}\,r}\left(0,\ 0,\ 1,\ -\frac{i}{\sin\theta}\right) \end{aligned} \tag{2}$$

Performing the following transformation $r' \to r + \imath a\cos\theta$ and $v' \to v - \imath a\cos\theta$ on tetrad leads to:

$$l'^a = \frac{\partial x'^a}{\partial r} l^r = \frac{1}{\sqrt{1+l}} \delta^a_r$$

$$n'^a = \frac{\partial x'^a}{\partial x^a} n^a = \delta^a_v - \frac{1}{2\sqrt{1+l}} \left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right) \delta^a_r$$

We will need following relation for transforming the tetrad m^a and $\bar{m}^a$

$$\frac{\partial r'}{\partial \theta} = \frac{\partial}{\partial \theta}(r + \imath a \cos\theta) = -\imath a \sin\theta \tag{3}$$

$$\frac{\partial v'}{\partial \theta} = \frac{\partial}{\partial \theta}(v - \imath a \cos\theta) = \imath a \sin\theta \tag{4}$$

Now, moving onto the tetrads m^a and $\bar{m}^a$ which could be given as:

$$m^a = \frac{1}{\sqrt{2}r}\left(\delta^a_\theta + \frac{\imath}{\sin\theta}\delta^a_\phi\right) \qquad \text{and} \qquad \bar{m}^a = \frac{1}{\sqrt{2}\bar{r}}\left(\delta^a_\theta - \frac{\imath}{\sin\theta}\delta^a_\phi\right)$$

Transforming them leads to the following form:

$$\begin{aligned}
m'^a &= \frac{\partial x'^a}{\partial x^a} m^a = \frac{\partial x'^a}{\partial x^\theta} m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= \left(\frac{\partial x'^a}{\partial r'}\frac{\partial r'}{\partial \theta} + \frac{\partial x'^a}{\partial v'}\frac{\partial v'}{\partial \theta} + \frac{\partial x'^a}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\right) m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= \left(-\imath a \sin\theta \delta^a_{r'} + \imath a \sin\theta \delta^a_{v'} + \delta^a_\theta\right) m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= \frac{1}{\sqrt{2}r}\left(\delta^a_\theta + (\delta^a_{v'} - \delta^a_{r'})\imath a \sin\theta + \frac{\imath}{\sin\theta}\delta^a_\phi\right) && \left(\text{using } m^\theta = \tfrac{1}{\sqrt{2}r}\delta^\theta_\theta\right) \\
&= \frac{1}{\sqrt{2}(r' - \imath a\cos\theta)}\left(\delta^a_\theta + (\delta^a_{v'} - \delta^a_{r'})\imath a \sin\theta + \frac{\imath}{\sin\theta}\delta^a_\phi\right)
\end{aligned}$$

$$\begin{aligned}
\bar{m'}^a &= \frac{\partial x'^a}{\partial x^a} \bar{m}^a = \frac{\partial x'^a}{\partial x^\theta} \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= \left(\frac{\partial x'^a}{\partial \bar{r}'}\frac{\partial \bar{r}'}{\partial \theta} + \frac{\partial x'^a}{\partial \bar{v}'}\frac{\partial \bar{v}'}{\partial \theta} + \frac{\partial x'^a}{\partial \theta'}\frac{\partial \theta'}{\partial \theta}\right) \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= \left(\imath a \sin\theta \delta^a_{r'} - \imath a \sin\theta \delta^a_{v} + \delta^a_\theta\right) \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= \frac{1}{\sqrt{2}\bar{r}}\left(\delta^a_\theta - (\delta^a_{v} - \delta^a_{r'})\imath a \sin\theta - \frac{\imath}{\sin\theta}\delta^a_\phi\right) && \left(\text{using } \bar{m}^\theta = \tfrac{1}{\sqrt{2}\bar{r}}\delta^\theta_\theta\right) \\
&= \frac{1}{\sqrt{2}(r' + \imath a\cos\theta)}\left(\delta^a_\theta - (\delta^a_{v} - \delta^a_{r'})\imath a \sin\theta - \frac{\imath}{\sin\theta}\delta^a_\phi\right)
\end{aligned}$$

After having calculated the relevent tetrads by Newman Janis algorithm, we will now proceed to construct the metric tensor which takes the following:

$$g^{\mu\nu} = \begin{vmatrix}
\frac{a^2\sin^2\theta}{r'^2+a^2\cos^2\theta} & -\frac{1}{\sqrt{1+l}} - \frac{a^2\sin^2\theta}{r'^2+a^2\cos^2\theta} & 0 & \frac{a}{r'^2+a^2\cos^2\theta} \\
-\frac{1}{\sqrt{1+l}} - \frac{a^2\sin^2\theta}{r'^2+a^2\cos^2\theta} & \frac{\left(1-\frac{2Mr}{r'^2+a^2\cos^2\theta}\right)}{1+l} + \frac{a^2\sin^2\theta}{r'^2+a^2\cos^2\theta} & 0 & -\frac{a}{r'^2+a^2\cos^2\theta} \\
0 & 0 & \frac{1}{(r'^2+a^2\cos^2\theta)} & 0 \\
\frac{a}{r'^2+a^2\cos^2\theta} & -\frac{a}{r'^2+a^2\cos^2\theta} & 0 & \frac{1}{(r'^2+a^2\cos^2\theta)\sin^2\theta}
\end{vmatrix}$$

The inverse is given as:

$$g_{\mu\nu} = \begin{vmatrix} -\left(1-\frac{2Mr}{\rho^2}\right) & -\sqrt{1+l} & 0 & a\sin^2\theta\left[\left(1-\frac{2Mr}{\rho^2}\right)-\sqrt{1+l}\right] \\ -\sqrt{1+l} & 0 & 0 & a\sqrt{1+l}\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ a\sin^2\theta\left[\left(1-\frac{2Mr}{\rho^2}\right)-\sqrt{1+l}\right] & a\sqrt{1+l}\sin^2\theta & 0 & \left[r^2+a^2(\cos^2\theta+\sqrt{1+l}\sin^2\theta)-ag_{t\phi}\right]\sin^2\theta \end{vmatrix}$$

where we defined $\rho^2 = r^2 + a^2\cos^2\theta$. From this metric, we can write the spacetime interval as follows:

$$\begin{aligned}
ds^2 &= -fdv^2 - 2\sqrt{1+l}dvdr + 2a\sin^2\theta(f-\sqrt{1+l})dvd\phi + 2a\sqrt{1+l}\sin^2\theta drd\phi \\
&\quad \rho^2dr^2 + [r^2 + a^2(\cos^2\theta + \sqrt{1+l}\sin^2\theta) + a^2\sin^2\theta(\sqrt{1+l}-f)]\sin^2\theta d\phi^2 \\
&= -f(dv - a\sin^2\theta d\phi)^2 - 2\sqrt{1+l}(dv - a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi) + \rho^2 d\Omega^2 \\
&= -f[dt + g(r)dr - a\sin^2\theta d\phi - a\sin^2\theta h(r)dr]^2 - 2\sqrt{1+l}[dt + g(r)dr \\
&\quad - a\sin^2\theta d\phi - a\sin^2\theta h(r)dr][dr + a\sin^2\theta d\phi + a\sin^2\theta h(r)dr] + \rho^2 d\Omega^2 \\
&= -f\left[dt + \frac{\rho^2}{\Delta}dr - a\sin^2\theta d\phi\right]^2 - 2\sqrt{1+l}\left[dt + \frac{\rho^2}{\Delta}dr - a\sin^2\theta d\phi\right] \\
&\quad \times [(1 + a\sin^2\theta h(r))dr + a\sin^2\theta d\phi] + \rho^2 d\Omega^2 \qquad \text{(using } \rho/\Delta = g(r) - a\sin^2\theta h(r)\text{)} \\
&= -fdt^2 - f\frac{\rho^4}{\Delta^2}dr^2 - fa^2\sin^4\theta d\phi^2 - 2f\frac{\rho^2}{\Delta}dtdr + 2af\frac{\rho^2}{\Delta}\sin^2\theta drd\phi + 2af\sin^2\theta dtd\phi \\
&\quad - 2\sqrt{1+l}\left(1 + ah(r)\sin^2\theta\right)dtdr - 2\sqrt{1+l}\frac{\rho^2}{\Delta}(1 + a\sin^2\theta h(r))dr^2 \\
&\quad + 2a\sqrt{1+l}\sin^2\theta[1 + a\sin^2\theta h(r)]d\phi dr - 2a\sqrt{1+l}\sin^2\theta dtd\phi - 2\sqrt{1+l}\frac{\rho^2 a\sin^2\theta}{\Delta}d\phi dr \\
&\quad + 2a^2\sqrt{1+l}\sin^4 d\phi^2 + \rho^2 d\theta^2 + h^2\rho^2\sin^2\theta dr^2 + \rho^2\sin^2\theta d\phi^2 + 2h\rho^2\sin^2\theta drd\phi \\
&= -fdt^2 - \left[f\frac{\rho^4}{\Delta^2} - h^2\rho^2\sin^2\theta + 2\sqrt{1+l}\frac{\rho^2}{\Delta}(1 + a\sin^2\theta h(r))\right]dr^2 + 2a\sin^2\theta(f - \sqrt{1+l})dtd\phi \\
&\quad + \left[2h\rho^2\sin^2\theta + 2af\frac{\rho^2}{\Delta}\sin^2\theta + 2a\sqrt{1+l}\sin^2\theta(1 + a\sin^2\theta h(r)) - 2a\sqrt{1+l}\frac{\rho^2\sin^2\theta}{\Delta}\right]drd\phi \\
&\quad - 2\left[f\frac{\rho^2}{\Delta} + \sqrt{1+l}(1 + ah(r)\sin^2\theta)\right]dtdr + \rho^2 d\theta^2 + [r^2 + a^2(\cos^2\theta + \sqrt{1+l}\sin^2\theta) \\
&\quad + a^2\sin^2\theta(\sqrt{1+l} - f)]\sin^2\theta d\phi^2 \\
&= -fdt^2 - g_{rr}dr^2 + [\rho^2 + a^2\sqrt{1+l}\sin^2\theta + a^2\sin^2\theta(\sqrt{1+l} - f)]\sin^2\theta d\phi^2 + \rho^2 d\theta + g_{tr}dtdr \\
&\quad + g_{r\phi}drd\phi + 2a\sin^2\theta(f - \sqrt{1+l})dtd\phi
\end{aligned}$$

Thus, imposing the condition, $g_{tr} = 0$ and $g_{r\phi} = 0$

$$\begin{aligned}
g_{r\phi} &= 0 \\
\implies h(r) &= -\frac{a(f\rho^2 + \sqrt{1+l}(\Delta - \rho^2))}{\Delta(\rho^2 + a^2\sqrt{1+l}\sin^2\theta)}
\end{aligned}$$

and

$$g_{tr} = 0$$
$$\implies h(r) = -\frac{\sqrt{1+l}\Delta + f\rho^2}{a\sqrt{1+l}\Delta \sin^2\theta}$$

Comparing both expressions, we thus have:

$$\frac{\sqrt{1+l}\Delta + f\rho^2}{a\sqrt{1+l}\Delta \sin^2\theta} = \frac{a(f\rho^2 + \sqrt{1+l}(\Delta - \rho^2))}{\Delta(\rho^2 + a^2\sqrt{1+l}\sin^2\theta)}$$
$$(\rho^2 + a^2\sqrt{1+l}\sin^2\theta)(\sqrt{1+l}\Delta + f\rho^2) = a^2[f\rho^2 + \sqrt{1+l}(\Delta - \rho^2)]\sqrt{1+l}\sin^2\theta$$
$$\Delta = -\frac{f\rho^2 + a^2(1+l)\sin^2\theta}{\sqrt{1+l}}$$
$$\sqrt{1+l}\Delta + f\rho^2 = -a^2(1+l)\sin^2\theta \tag{5}$$

$$\implies h(r) = \frac{a\sqrt{1+l}}{\Delta} \tag{6}$$

The radial component of metric tensor is given as:

$$\begin{aligned} g_{rr} &= f\frac{\rho^4}{\Delta^2} - h(r)^2\rho^2\sin^2\theta + 2\sqrt{1+l}\frac{\rho^2}{\Delta}[1 + a\sin^2\theta h(r)] \\ &= \frac{\rho^2}{\Delta}\left[-h(r)^2\Delta\sin^2\theta + \sqrt{1+l}\{1 + a\sin^2\theta h(r)\}\right] && \text{(using } g_{tr} = 0\text{)} \\ &= \frac{\rho^2}{\Delta}\left[\sqrt{1+l} - h(r)^2\Delta\sin^2\theta + ah(r)\sqrt{1+l}\sin^2\theta\right] \\ &= \sqrt{1+l}\frac{\rho^2}{\Delta} && \text{(using 6)} \end{aligned}$$

Thus the final expression for spacetime interval:

$$\begin{aligned} ds^2 = &-fdt^2 - \sqrt{1+l}\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + 2a\sin^2\theta(f - \sqrt{1+l})dtd\phi \\ &+ [\rho^2 + a^2\sqrt{1+l}\sin^2\theta + a^2\sin^2\theta(\sqrt{1+l} - f)]\sin^2\theta d\phi^2 \end{aligned}$$

$$\begin{aligned} = &-fdt^2 + \frac{\rho^2(1+l)}{f\rho^2 + a^2(1+l)\sin^2\theta}dr^2 + 2a\sin^2\theta(f - \sqrt{1+l})dtd\phi \\ &+ \rho^2 d\theta^2 + [\rho^2 + a^2\sin^2\theta\sqrt{1+l} + a^2\sin^2\theta(\sqrt{1+l} - f)]\sin^2\theta d\phi^2 \end{aligned}$$

Finally the metric has been completely fixed following the Newmann Janis algorithm.

$$g_{\mu\nu} = \begin{vmatrix} -f & 0 & 0 & a\sin^2\theta(f - \sqrt{1+l}) \\ 0 & \frac{\rho^2\sqrt{1+l}}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ a\sin^2\theta(f - \sqrt{1+l}) & 0 & 0 & \frac{\Sigma}{\rho^2}\sin^2\theta \end{vmatrix}$$

where

$$\Sigma = (r^2 + a^2)^2 - a^2\sqrt{1+l}\Delta\sin^2\theta + a^4 l\sin^4\theta + 2\rho^2 a^2\sin^2\theta(\sqrt{1+l} - 1)$$

with

$$f = 1 - \frac{2Mr}{\rho^2}$$

and

$$\begin{aligned}\sqrt{1+l}\Delta &= f\rho^2 + a^2(1+l)\sin^2\theta \\ &= -2Mr + \rho^2 + a^2(1+l)\sin^2\theta \\ &= r^2 + a^2 - 2Mr + a^2 l \sin^2\theta\end{aligned}$$

where we used:

$$\rho^2 = r^2 + a^2\cos^2\theta$$

In this metric we have event horizons at $\Delta = 0 \to r = M \pm \sqrt{M^2 - a^2(1+l)\sin^2\theta}$ whereas the ergosphere lies at $\rho^2 - 2Mr = 0 \to r = M \pm \sqrt{M^2 - a^2\cos^2\theta}$. On direct comparison with [10] we see that the $g_{t\phi}$ is different. However this is so because the bumblebee field used in Einstein Field Equation was defined to be $b_\mu = \left(0, b_0\sqrt{\frac{1+l}{1-2M/r}}, 0, 0\right)$. However, the Newman Janis Algorithm was performed on metric tensor where $b_\mu = (0, b_0, 0, 0)$. In the limit $a \to 0$, we recover the stationary solution.

$$ds^2 = -fdt^2 + \frac{1+l}{f}dr^2 + r^2 d\Omega^2$$

with $f = 1 - 2M/r$. However an interesting remark to be made here is that under $a^2 \to 0$, we have:

$$ds^2 = -fdt^2 + \frac{1+l}{f}dr^2 + 2a\sin^2\theta(f - \sqrt{1+l})dtd\phi + r^2 d\Omega^2$$

This result differs significantly from the one described in [10]. However it can be recovered using binomial approximation and the limit $l/2 \to 0$.

$$ds^2 = -fdt^2 + \frac{1+l}{f}dr^2 + 2a\sin^2\theta(f - 1)dtd\phi + r^2 d\Omega^2$$

The inverse metric tensor takes this form:

$$g^{\mu\nu} = \begin{vmatrix} -\frac{\Sigma}{\rho^2\Delta\sqrt{1+l}} & 0 & 0 & -a\frac{\sqrt{1+l}-f}{\Delta\sqrt{1+l}} \\ 0 & \frac{\Delta}{\rho^2\sqrt{1+l}} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ -a\frac{\sqrt{1+l}-f}{\Delta\sqrt{1+l}} & 0 & 0 & \frac{f\rho^2}{\rho^2\Delta\sqrt{1+l}\sin^2\theta} \end{vmatrix}$$

Here $g^{t\phi} = -a\frac{\sqrt{1+l}-f}{\Delta\sqrt{1+l}} = -\frac{a}{\rho^2\Delta\sqrt{1+l}}[r^2 + a^2 + a^2 l\sin^2\theta - \sqrt{1+l}\Delta + (\sqrt{1+l} - 1)\rho^2]$

3 Shadow Calculation

There are several aspect of the new geometry that one could study in principle. The way photons move around the Black Hole emersed in Lorentz violating Bumblebee background field has direct observational signature in Black Hole shadow. In this paper we will resort to studying the null geodesics Using Hamilton Jacobi Equation. The Hamiltonian in General Relativity is defined as:

$$H(x^\mu, p_\mu, \lambda) = \frac{1}{2}p_\mu p^\mu = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu.$$

which could be used to write the Hamilton-Jacobi equation in following manner:

$$H\left(x^\mu, \frac{\partial S}{\partial x^\mu}, \lambda\right) = -\frac{\partial S}{\partial \lambda},$$

which leads to:

$$\frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = -\frac{\partial S}{\partial \lambda}.$$

$$\frac{1}{2}g^{tt}[\partial_t S]^2 + \frac{1}{2}g^{t\varphi}\,\partial_t S\,\partial_\varphi S - \frac{1}{2}g^{rr}[\partial_r S]^2 + \frac{1}{2}g^{\theta\theta}[\partial_\theta S]^2 + \frac{1}{2}g^{\varphi\varphi}[\partial_\varphi S]^2 = -\partial_\lambda S.$$

Here we have utilized the Kerr symmetry which ensures that the Hamiltonian does not explicitly depend on the coordinates t, φ, or the parameter λ. So, the general form of the action could be written as following:

$$S(x^\mu; \alpha_\mu; \lambda) = p_t t + p_\varphi \varphi + \gamma\lambda + S_2(r, \theta),$$

where

$$\frac{\partial S}{\partial \lambda} = \gamma = -H = \frac{\mu^2}{2}.$$

The constants of motion $E \equiv -p_t$ and $\Phi \equiv p_\varphi$ correspond to the energy and the angular momentum along the z-axis, respectively, measured at spatial infinity. For rest of the paper we will use natural units, where $c = 1$ and $G = 1$.

We now assume that the function $S_2(r, \theta)$ can be separated into two parts:

$$S_2(r, \theta) = S_r(r) + S_\theta(\theta).$$

If S is constructed using this ansatz and satisfies the Hamilton-Jacobi equation, then it should correctly describe the particle's motion. According to this form, the action function S becomes:

$$S(x^\mu; \alpha_\mu; \lambda) = \frac{\mu^2}{2}\lambda - Et + \Phi\varphi + S_r(r) + S_\theta(\theta) \tag{7}$$

Substituting this into the Hamilton-Jacobi equation gives:

$$\underbrace{g^{tt}E^2 + g^{\phi\phi}\Phi - 2g^{t\phi}E\Phi}_{\mathcal{C}} + g^{rr}\left(\frac{\partial S_r}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = -\mu^2$$

It can be be further simplified:

$$\mathcal{C} = -\frac{\Sigma}{\rho^2\Delta\sqrt{1+l}}E^2 + \frac{\sqrt{1+l}\Delta - a^2(1+l)\sin^2\theta}{\rho^2\Delta\sqrt{1+l}\sin^2\theta}\Phi^2 + 2a\frac{\sqrt{1+l} - f}{\Delta\sqrt{1+l}}E\Phi$$

$$= \frac{-1}{\rho^2\Delta\sqrt{1+l}}\left(\rho^4E^2 + a^2(1+l)\Phi^2 - 2\rho^2a\sqrt{1+l}E\Phi\right) + \frac{a^2\rho^2\sin^2\theta(f - 2\sqrt{1+l})}{\rho^2\Delta\sqrt{1+l}}E^2 + \dots$$

$$= \frac{-1}{\rho^2\Delta\sqrt{1+l}}(\rho^2E - a\sqrt{1+l}\Phi)^2 + \frac{a^2\sin^2\theta f}{\Delta\sqrt{1+l}}E^2 + \frac{\Phi^2}{\rho^2\sin^2\theta} - 2\frac{af}{\Delta\sqrt{1+l}}E\Phi - 2\frac{a^2\sin^2\theta}{\Delta}E^2$$

$$= \frac{-1}{\rho^2\Delta\sqrt{1+l}}(\rho^2E - a\sqrt{1+l}\Phi)^2 + \frac{a^2\sin^2\theta}{\rho^2}E^2 + \frac{\Phi^2}{\rho^2\sin^2\theta} - 2\frac{afE\Phi}{\Delta\sqrt{1+l}} - a^2\sin^2\theta\frac{2\rho^2 + a^2\sqrt{1+l}\sin^2\theta}{\rho^2\Delta}E^2$$

$$= \frac{-1}{\rho^2\Delta\sqrt{1+l}}(\rho^2E - a\sqrt{1+l}\Phi)^2 + \frac{a^2\sin^2\theta}{\rho^2}E^2 + \frac{\Phi^2}{\rho^2\sin^2\theta} - \frac{2aE\Phi}{\rho^2} + 2\frac{a\sqrt{1+l}\Phi}{\rho^2\Delta}a^2E\sin^2\theta$$
$$- a^2E^2\sin^2\theta\frac{2\rho^2 + a^2\sqrt{1+l}\sin^2\theta}{\rho^2\Delta}$$

$$= \frac{-1}{\rho^2\Delta\sqrt{1+l}}(\rho^2 E - a\sqrt{1+l}\Phi)^2 + \frac{1}{\rho^2\sin^2\theta}[aE\sin^2\theta - \Phi]^2 + \frac{2a^2E\sin^2\theta}{\rho^2\Delta}a\sqrt{1+l}\Phi$$
$$- \frac{a^2E^2\sin^2\theta}{\rho^2\Delta}(2\rho^2 + a^2\sqrt{1+l}\sin^2\theta)$$

$$= -\frac{1}{\rho^2\Delta\sqrt{1+l}}(\rho^2 E - a\sqrt{1+l}\Phi)^2 + \frac{1}{\rho^2\sin^2\theta}[aE\sin^2\theta - \Phi]^2 - \frac{a^4(1+l)\sin^4\theta E^2}{\rho^2\Delta\sqrt{1+l}}$$
$$- \frac{2a^2E}{\rho^2\Delta}(E\rho^2 - a\sqrt{1+l}\Phi)\sin^2\theta$$

$$= -\frac{1}{\rho^2\Delta\sqrt{1+l}}\underbrace{(E(r^2+a^2) - a\sqrt{1+l}\Phi)^2}_{P^2} + \frac{1}{\rho^2\sin^2\theta}[aE\sin^2\theta - \Phi]^2 - l\frac{a^4\sin^4\theta E^2}{\rho^2\Delta\sqrt{1+l}}$$
$$- \frac{2a^2E}{\rho^2\Delta}(E\rho^2 - a\sqrt{1+l}\Phi)\sin^2\theta\left[1 - \frac{1}{\sqrt{1+l}}\right]$$

Which leads to following useful form;

$$\mathcal{C} + \frac{\Delta}{\rho^2\sqrt{1+l}}p_r^2 + \frac{1}{\rho^2}p_\theta^2 = -\mu^2$$
$$-\frac{P^2}{\rho^2\Delta\sqrt{1+l}} + \frac{1}{\rho^2\sin^2\theta}[aE\sin^2\theta - \Phi]^2 - l\frac{a^4\sin^4\theta E^2}{\rho^2\Delta\sqrt{1+l}}$$
$$-\frac{2a^2E}{\rho^2\Delta}(E\rho^2 - a\sqrt{1+l}\Phi)\sin^2\theta\left[1 - \frac{1}{\sqrt{1+l}}\right] + \frac{\Delta}{\rho^2\sqrt{1+l}}p_r^2 + \frac{1}{\rho^2}p_\theta^2 = -\mu^2$$

scaling by $\rho^2\Delta\sqrt{1+l}\sin^2\theta$

$$- P^2\sin^2\theta + \Delta\sqrt{1+l}[aE\sin^2\theta - \Phi]^2 - la^4\sin^6\theta E^2 - 2a^2E(E\rho^2 -$$
$$a\sqrt{1+l}\Phi)\sin^4\theta(\sqrt{1+l} - 1) + \Delta^2p_r^2\sin^2\theta + p_\theta^2\Delta\sqrt{1+l}\sin^2\theta$$
$$= -\rho^2\Delta\mu^2\sqrt{1+l}\sin^2\theta$$

$$-P^2\sin^2\theta + \Delta^2p_r^2\sin^2\theta + r^2\Delta\mu^2\sqrt{1+l}\sin^2\theta - 2a^2E^2r^2\sin^4\theta(\sqrt{1+l} - 1) =$$
$$- \Delta\sqrt{1+l}[aE\sin^2\theta - \Phi]^2 + la^4\sin^6\theta E^2 + 2a^2E[Ea^2\cos^2\theta - a\sqrt{1+l}\Phi]\sin^4\theta(\sqrt{1+l} - 1)$$
$$- p_\theta^2\Delta\sqrt{1+l}\sin^2\theta - a^2\Delta\mu^2\sqrt{1+l}\cos^2\theta\sin^2\theta$$

$$(-P^2 + \Delta^2p_r^2 + r^2\Delta\mu^2\sqrt{1+l})\sin^2\theta - \underbrace{2a^2E^2r^2\sin^4\theta(\sqrt{1+l} - 1)}_{\text{Extra term}} = -\Delta\sqrt{1+l}[(aE\sin^2\theta - \Phi)^2$$
$$+ p_\theta^2\sin^2\theta + a^2\mu^2\cos^2\theta\sin^2\theta] + \underbrace{la^4\sin^6\theta E^2 + 2a^2E[Ea^2\cos^2\theta - a\sqrt{1+l}\Phi]\sin^4\theta(\sqrt{1+l} - 1)}_{\text{Extra term}}$$

It can be seen that the system can not be broken down via separation of variables due to the presence of l. Thus, we use the limit $a^2l \to 0$ with binomial approximation on $\sqrt{1+l} - 1 \approxeq -\frac{l}{2}$ for radial and angular motion. We divided both sides by $\sin^2\theta\Delta\sqrt{1+l}$, thus we have:

$$\frac{-P^2 + \Delta^2p_r^2 + r^2\Delta\mu^2\sqrt{1+l}}{\Delta\sqrt{1+l}} = -\left[\left(aE\sin\theta - \frac{\Phi}{\sin\theta}\right)^2 + p_\theta^2 + a^2\mu^2\cos^2\theta\right]$$

It is seen from above that only the radial motion is affected due to the presence of bumblebee field. Thus, the Carter constant should need no modification.

$$Q \equiv K - (aE - \Phi)^2$$

Thus, we have

$$\Delta^2 p_r^2 = P^2 - \Delta\sqrt{1+l}[K + r^2\mu^2]$$

and $\dot{r} = p^r = g^{rr}p_r = \frac{\Delta}{\rho^2\sqrt{1+l}}p_r$:

$$\sqrt{1+l}\rho^2\dot{r} = \pm\sqrt{R} \quad \text{with} \quad R = P^2 - \Delta\sqrt{1+l}[Q + (aE - \Phi)^2 + r^2\mu^2]$$

$$\text{and} \quad P^2 = [E(a^2 + r^2) - a\sqrt{1+l}\Phi]^2 \tag{8}$$

The θ motion is exactly the same due to the approximation made:

$$\rho^2\dot{\theta} = \pm\sqrt{\Theta} \quad \text{with} \quad \Theta = Q - \cos^2\theta\left[a^2(\mu^2 - E^2) + \frac{\Phi^2}{\sin^2\theta}\right] \tag{9}$$

Motion is t and ϕ is given as

$$p^t = g^{tt}p_t + g^{t\phi}p_\phi$$

Thus

$$\rho^2\dot{t} = \frac{\Sigma}{\Delta\sqrt{1+l}}E - a\frac{(\sqrt{1+l}-1)\rho^2 + 2Mr}{\Delta\sqrt{1+l}}\Phi \tag{10}$$

and

$$p^\phi = g^{\phi t}p_t + g^{\phi\phi}p_\phi \tag{11}$$

therefore;

$$\rho^2\dot{\phi} = a\frac{(\sqrt{1+l}-1)\rho^2 + 2Mr}{\Delta\sqrt{1+l}}E + \frac{\sqrt{1+l}\Delta - a^2(1+l)\sin^2\theta}{\Delta\sqrt{1+l}\sin^2\theta}\Phi \tag{12}$$

4 Spherical Orbit of Photon and Shadow

The condition for the spherical orbit of constant radial coordinate r, are:

$$\dot{r} = 0 \qquad \ddot{r} = 0$$

which means that the null radial velocity and null radial acceleration should vanish. Before we proceed further, we define two parameters

$$\xi = \frac{\Phi}{E} \qquad \eta = \frac{Q}{E^2}$$

and consider following redefinition for photons ($\mu = 0$):

$$\mathcal{R}(r) = \frac{R(r)}{E^2} = [(r^2 + a^2) - a\xi\sqrt{1+l}]^2 - \Delta\sqrt{1+l}[\eta + (a - \xi)^2]$$

From (8), it is clear that condition (11) translates into:

$$\mathcal{R}(r) = 0 \qquad \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}r} = 0$$

Hence

$$\xi = \frac{-4r\Delta + a^2\Delta' + r^2\Delta'}{a\sqrt{1+l}\Delta'(r)}$$

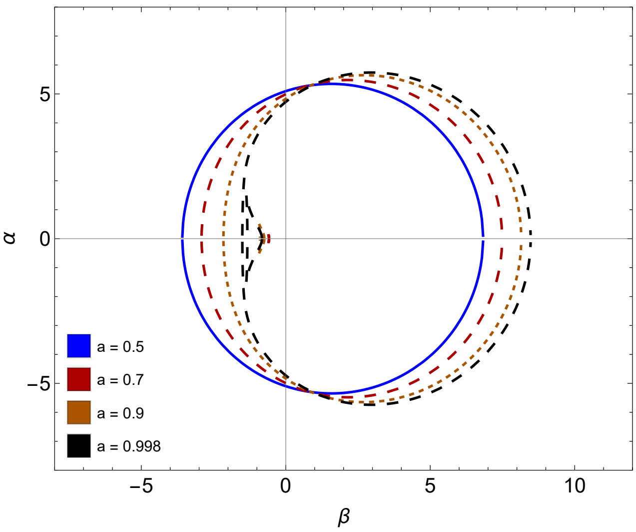

Figure 1: The Shape of photon sphere with varying angular momenta as parameter for $l = 1$ is plotted.

$$\eta = \frac{-16r^2\sqrt{1+l}\Delta^2(r) + 16a^2r^2(1+l)\Delta(r) + 8r[r^2 - a^2(\sqrt{1+l}-1)]\sqrt{1+l}\Delta(r)\Delta'(r)}{a^2(1+l)^{3/2}\Delta'^2} + \frac{[a^4\{l(\sqrt{1+l}-2) + 2(\sqrt{1+l}-1)\} + 2a^2(\sqrt{1+l}-1-l)r^2 + r^4\sqrt{1+l}]\Delta'^2(r)}{a^2(1+l)^{3/2}\Delta'^2}$$

where

$$\Delta = \frac{r^2 + a^2 - 2Mr + a^2 l \sin^2\theta}{\sqrt{1+l}} \approx \frac{r^2 + a^2 - 2Mr}{\sqrt{1+l}}$$

represents the photon sphere.

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